

# Eigenvalue problems for the complex PT-symmetric Potential

$$V(x) = igx$$

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## Abstract

The spectrum of complex PT-symmetric potential,  $V(x) = igx$ , is known to be null. We enclose this potential in a hard-box:  $V(|x| \geq 1) = \infty$  and in a soft-box:  $V(|x| \geq 1) = 0$ . In the former case, we find real discrete spectrum and the exceptional points of the potential. The asymptotic eigenvalues behave as  $E_n \sim n^2$ . The solvable purely imaginary PT-symmetric potentials vanishing asymptotically known so far do not have real discrete spectrum. Our solvable soft-box potential possesses two real negative discrete eigenvalues if  $|g| < (1.22330447)$ . The soft-box potential turns out to be a scattering potential not possessing reflectionless states.

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A complex Hamiltonian that does not change under the joint transformation of Parity (P:  $x \rightarrow -x$ ) and Time-reversal ( $i \rightarrow -i$ ) is called PT-symmetric:  $(PT)H(PT)^{-1} = H$  or  $[PT, H] = 0$ . When two (linear) operators A and B commute, we can arrange for simultaneous eigenstates. However, if one of the operators is anti-linear:  $A(c\psi) = c^*\psi^*$ , then for A and B we will have simultaneous eigenstates if only the eigenvalues of B are real. In case of complex eigenvalues of B, the simultaneity of eigenstates does not hold. Further, it follows that B can have real or complex conjugate pairs of eigenvalues. Therefore, PT being anti linear a Hamiltonian commuting with it may have entire discrete spectrum as real or as complex conjugate pairs or both. In the first case PT-symmetry is unbroken or exact in the second case it breaks down spontaneously.

The revelation that a complex PT-symmetric potential [1] too may possess real discrete

spectrum has lead to very interesting developments [2]. In this regard, the exactly solvable potentials have been very revealing [3]. Quasi-exactly solvable complex PT-symmetric potentials [4] have also been helpful wherein one gets only a certain number of eigenvalues and eigenstates with some restriction on the parameters of the potential. Next some simple semi-analytically [5] solvable potentials have been proposed wherein one gets implicit equation for energy to calculate the discrete spectrum iteratively and numerically. These potentials have been useful and handy in some tedious calculations

For the potential  $V_B(x) = -(ix)^\nu$  [1], a few critical values of  $\nu$  in (1.42207,2) are known where the spectrum has a few real eigenvalues and for  $\nu \geq 2$  the entire spectrum is real. For analytically solvable potential,  $V_S(x) = -V_1 \text{sech}^2 x + iV_2 \text{sech} x \tanh x$ , [3] the discrete spectrum is real if  $|V_2|$  is less than or equal to a critical value  $\sqrt{V_1 + 1/4}$ . Otherwise the spectrum consists of complex conjugate pairs of energy. The semi-analytically solvable potential  $V_Z(|x| < 1) = iZ\theta(x)$ ,  $V_Z(|x| > 1) = \infty$  [5] has been found to have critical values of  $Z$  above which PT-symmetry is spontaneously broken. Here  $\theta(x \leq 0) = -1$  and  $\theta(x > 0) = 1$ .

Nowadays, with the advent of the packages like *Mathematica* one can perform quick and almost error-free calculations involving even the higher order functions. This gives us a broader scope to extend at least the class of semi-analytically solvable potentials. In this Letter, we present the complex PT-symmetric potential  $V(x) = igx$  in two modifications. These two models are semi-analytically solvable which also arise due to the following open questions in PT-symmetric quantum mechanics [6,7].

Another way of visualizing real spectrum of a non-Hermitian Hamiltonian is through the exceptional points of a Hamiltonian as proposed by Kato (1960) [8,9]. Let the eigenvalues  $E_n(\lambda)$ , of  $H_\lambda = H_1 + \lambda H_2$  be analytic function of  $\lambda$ , where  $H_1, H_2$  are Hermitian Hamiltonians. If  $\lambda$  is analytically continued say as  $\lambda = i\mu$  in a complex domain, then at certain values of  $\mu = \mu_c$  two real distinct eigenvalues and eigenstates may coalesce (merge) in to one. Further, if  $\mu > \mu_c$  the coalesced eigenvalues reappear as complex conjugate pair. In this complex extension of the parameter  $\lambda$  the Hamiltonian becomes non-Hermitian. We strongly feel that PT-symmetry or more generally pseudo-Hermiticity [10] ( $\eta H \eta^{-1} = H^\dagger$ ) may be the necessary condition for the existence of exceptional points in  $H_{i\mu}$ .

So far we do not know the necessary and sufficient condition to determine *a priori* whether a given complex PT-symmetric potential will have real discrete spectrum. However, case of  $V(x) = igx$  is well studied wherein using the arguments of Stokes and Anti-stokes lines the spectrum has been found to be null. An interesting *a priori* and *necessary* condition in

this regard is that the potential ought to have a pair of classical turning points (solutions of  $E = V(x)$ ) of the type  $\pm a + ib$  [11] at a given real energy. In addition to this, any singularity of the potential should not lie on the line from  $z = -a + ib$  to  $z = a + ib$ . This explains why the real discrete spectrum of  $V_L(x) = igx$  [1],  $V_E(x) = iV_2 \tanh x \operatorname{sech} x$ ,  $V_G(x) = iV_2 x e^{-x^2}$  and  $V_C(x) = iV_2 x / (1 + x^2)$  for *any real value of  $V_2$*  is empty. Since only  $V_E(x)$  among these potentials is analytically solvable having the property of shape-invariance, therefore an argument based on supersymmetry may be ruled out for the emptiness of the real discrete spectrum. However, when a real attractive potential of sufficient strength is added, these potentials [3] do possess real discrete spectrum (e.g.,  $V_{HO}(x) = x^2 + igx$ ).

In this work, we study the purely imaginary potential,  $V_L(x)$ , within a hard-box ( $(V(|x| > 1) = \infty)$  or a soft-box ( $(V(|x| \geq 1) = 0)$  for real discrete spectrum. The abovementioned potentials,  $V_G(x)$  and  $V_E(x)$  may also be seen as purely imaginary PT-symmetric potentials vanishing asymptotically and possessing the real discrete spectrum as null. Similarly, we find that simple purely imaginary potential  $V(x) = iV_0 \theta(x)$  placed within the soft box also has real discrete spectrum as null. This gives us an additional motivation to discuss  $V_L(x)$  placed within a soft box.

First we study

$$V(x) = igx, V(|x| \geq 1) = \infty. \quad (1)$$

Taking  $2m = 1 = \hbar^2$  in one-dimensional time-independent Schrödinger equation, we get the wave solution as  $\psi(x) = \mathcal{A}Ai(z) + \mathcal{B}Bi(z)$  with  $z = (E - igx)/q$ ,  $q = (g^2)^{1/3}$ . The Dirichlet boundary conditions that  $\psi(x = \pm 1) = 0$  and their simultaneous consistency gives rise to an implicit and transcendental equation for  $E$  as

$$Ai[(E + ig)/q]Bi[(E - ig)/q] - Ai[(E - ig)/q]Bi[(E + ig)/q] = 0 \quad (2)$$

Here  $Ai(z)$  and  $Bi(z)$  are the Airy functions. The eigenfunction is expressible as

$$\psi(x) = \mathcal{C}(E, g)Ai[(E + igx)/q] + Bi[(E + igx)/q], \quad -1 \leq x \leq 1, \quad (3)$$

where

$$-\frac{Bi[(E + ig)/q]}{Ai[(E + ig)/q]} = \mathcal{C}(E, g) = -\frac{Bi[(E - ig)/q]}{Ai[(E - ig)/q]}. \quad (4)$$

For the real branch of  $q$ , the Eq. (2) say  $f(E) = 0$  is a real equation of  $E$  on real line. Therefore as per the *fundamental theorem of algebra* the roots (eigenvalues) will be either

purely real or complex-conjugate pairs. In the former case,  $\psi(x)$  will also be an eigenstate of the anti-linear operator PT ( $x \rightarrow -x$  and  $i \rightarrow -i$ ). In the latter case,  $\psi(x)$  will no more be the eigenstates of PT and the PT-symmetry will be spontaneously broken. We employ *Mathematica* to perform the calculations. We find the first critical value  $g_1 = 12.3124556046$  of  $g$ . If  $|g| < g_1$ , each calculated eigenvalue (root of from Eq. (2)) is real and discrete. Then if  $|g| = g_1$ , we find that only the lowest pair of real discrete eigenvalues merge (coalesce) into one real eigenvalue to make way (when  $|g| > g_1$ ) for a single pair of complex conjugate eigenvalues signifying spontaneous breakdown of PT-symmetry. When  $|g| = g_1$ , we notice the merger of lowest real eigenvalues at  $E = 7.1086$ . A sudden jump discontinuity is observed in the ground level while crossing a critical value e.g.,  $E_G(12.31) = 7.0316$ ,  $E_G(12.32) = 21.7209$ . The first five eigenvalues for  $g = 12.31$  are 7.03165, 7.1848, 21.7217, 39.1884, 61.4929. On the other hand, the first five eigenvalues for  $g = 12.32$  are  $7.1097 \pm .1342i$ , 21.7209, 39.1880, 61.4926(61.6850). The number in bracket equals  $25\pi^2/4$ . The next critical values of  $g$  are  $g_2 \approx 53.18$ ,  $g_3 \approx 122.90$ , ... The first five eigenvalues for  $g = 53$  are  $16.4942 \pm 24.4311i$ , 29.7350, 31.6153, 58.2078 and for  $g = 54$  they are  $16.7009 \pm 25.0725i$ ,  $30.8513 \pm 1.9731i$  and 58.0801.

Thus each critical value of  $g$  indicates the onset of the removal of lowest pair of *real* discrete eigenvalues. Removal of only ground level of a potential from its supersymmetric partner potential has earlier been observed in supersymmetric quantum mechanics [12]. In contrast to this, a complex PT-symmetric potential having infinite real discrete spectrum manifests in the removal of the lowest pair of *real* eigenvalues as its parameter passes through several critical values. When this happens the ground state eigenvalue becomes discontinuous and we have  $\Delta(g_1) = [(E_G(|g| = g_1 + \epsilon) - E_G(|g| = g_1 - \epsilon))]$ . When the parameter  $g$  increases the net density of states of *real* eigenvalues reduces.

Seeing the potential (1) as a non-Hermitian PT-symmetric perturbation to the one dimensional box having  $E_n^0 = n^2\pi^2/4$  presents interesting features. We find that there are two remarkable features of the complex PT-symmetric perturbation (1) in contrast to its real Hermitian counterpart when  $V(|x| < 1) = gx$ . In the former case, as the value of  $g$  increases the lowest eigenvalues (approximately corresponding to  $E_n^0$ ) of the real discrete spectrum get removed reducing the number of *real* eigenvalues below a fixed energy  $E = E^*$ . On the other hand, in the Hermitian case, these lowest eigenvalues get perturbed and get pushed down to become negative however the number of levels below  $E = E^*$  does not change. Secondly, we find that the higher real eigenvalues behave as  $E_n \sim E_n^0 + \epsilon_1^2$  whereas in the Hermitian case they are like  $E_n = E_n^0 - \epsilon_2^2$ ,  $\epsilon_1, \epsilon_2$  are small real numbers.

We also find that there are bands of values of  $g$ , wherein for a fixed value of  $g$ , we get a continuous band of energies which satisfy the eigenvalue equation (2), however the corresponding eigenstates are null. The discrete eigenvalues do exist on the left and right of this energy-band. For example when  $g = 3.4$ , we get a continuous band of energies  $23.2 < E < 26.5$  wherein every energy is an “eigenvalue” with a null eigenvector. For  $g = 5.0i$ , we get such an energy band as  $32.8 < E < 34.2$ .

The discrete spectrum of the Hamiltonian with a hard-box potential can also be conveniently calculated by diagonalizing the matrix  $H_{r,s} = \langle r|H|s \rangle$ , where  $|r \rangle$  are the complete orthonormal eigenstates of the one-dimensional hard-box potential namely  $V_H(|x| < 1) = 0, V_H(|x| > 1) = \infty$ . We get for our hard-box potential

$$H_{r,s} = \frac{r^2\pi^2}{4}\delta_{r,s} + (-1)^{t/2} \frac{16igrs}{\pi^2(r^2 - s^2)^2} \left[ \frac{1 + (-1)^t}{2} \right], \quad t = r + s + 1, r, s = 1, N. \quad (5)$$

Here  $N$  denotes the truncation or the order of the “infinite” dimensional matrix  $H_{r,s}$ . The eigenvalue determinant  $\det|H - EI|_{N \times N}$  is a very fast converging function of  $N$ . Consequently, the critical values of  $g$  discussed above hardly depend upon  $N$ . This in turn also means that first  $N$  eigenvalues (real or complex) calculated by fixing a value of  $g$  and the dimensions of the  $H$  matrix  $N \times N$  would be quite accurate. We would like to remark that the method of diagonalization of eigenvalues supplements the calculations using Eq. (2) specially when eigenvalues are complex conjugate pair.

The PT-symmetry is also conceived as P-pseudo-Hermiticity [9], it is therefore worth pointing that the matrix  $H_{r,s}$  (5) for the PT-symmetric matrix is pseudo-Hermitian for any value of  $g$  with parity operator given by  $P = (-1)^r \delta_{r,s}$ . Other property of  $H_{r,s}$  is that it is complex symmetric matrix. The experience gained in the diagonalization of a hard-box Hamiltonian is markedly different from that of diagonalization of the well known Hamiltonian of D. Bessis  $H_{DB} = p^2 - igx^3$  [1].

The diagonalization of  $H_{DB}$  in the real harmonic oscillator basis by truncating the dimensions of the matrix to finite:  $N \times N$ , contrary to yielding the whole discrete spectrum as real for any value of  $g$  [1], yields [13] a finite number ( $N$ ) of eigenvalues as real if  $|g| \geq g_c(N)$ . Here  $g_c(N)$  is crucially decreasing function of  $N$ . So eventually as  $N \rightarrow \infty$ ,  $g_c(N)$  vanishes justifying entire spectrum to be real for any real value of  $g$ .

The complex PT-symmetric Potential

$$V_R(|x| < 1) = -V_1 + iV_2, V_R(|x| > 1) = 0 \quad (6)$$

possesses scattering states for  $E > 0$  and bound states for  $E < 0$ . The interesting scattering

properties of (6) have earlier been studied for this potential [14]. We further find that when  $V_1 = 0$ , the purely imaginary potential (6) has the real discrete spectrum as empty. This potential along with  $V_E(x)$  and  $V_G(x)$  [3] constitute purely imaginary and asymptotically vanishing PT-symmetric potentials that do not possess any real discrete eigenvalue. We now study the second modification of  $igx$ -potential

$$V(|x| < 1) = igx, V(|x| > 1) = 0, \quad (7)$$

the soft-box potential that is exceptional in this regard. Assuming  $2m = 1 = \hbar^2$ , we consider the Schrödinger equation for (7) when  $E < 0$ . Using the Airy functions again, we write the wave solution of Schrödinger equation in three regions as  $\psi(x < -1) = ce^{\kappa x}$ ,  $\psi(-1 \leq x \leq 1) = aAi(z) + bBi(z)$ ,  $\psi(x > 1) = de^{-\kappa x}$ . Matching the wave solution at  $x = \pm 1$  and eliminating  $a, b, c, d$ , we get

$$[\kappa q Ai(s) + ig Ai'(s)][\kappa q Bi(t) - ig Bi'(t)] = [\kappa q Bi(s) + ig Bi'(s)][\kappa q Ai(t) - ig Ai'(t)] \quad (8)$$

Here  $s, t = (E \pm ig)/q$ ,  $z = (E - igx)/q$ ,  $q = (g^2)^{1/3}$  and  $\kappa = \sqrt{-E}$ . This equation is an implicit equation of  $-E$  which is real for real negative values of  $E$ . So as per the *fundamental theorem of Algebra* it will have either negative real roots or complex conjugate roots. By the evaluation of the roots of Eq. (8), we find that if  $|g| < 1.223830447 = g_c$  there exist two negative real discrete eigenvalues. Further if  $|g| = g_c$ , we find that two real discrete eigenvalues merge at  $E = .24994$ . For the values of  $|g| > g_c$ , the eigenvalues are complex conjugate pairs. As  $|g|$  is decreased one level is pushed down to negative values, the other one shifts closer to zero. For instance for  $|g| = 1.2$ ,  $E_1 = -0.40891$ ,  $E_2 = -.14426$ . For  $|g| = 0.6$ , we get  $E_1 = -2.58012$ ,  $E_2 = -0.00275$ . When  $|g| = 0.1$ , we get  $E_1 = -9.29466$ ,  $E_2 = -1.7849 \times 10^{-6}$ . We conjecture that the PT-symmetric potentials which have both bound and scattering states will have only one critical value of parameter as found here for (7) and in the models in Refs. [3].

The limit when  $g$  approaches zero is not transparent here. For this we use the asymptotics or the Airy functions  $Ai(s), Bi(s) \sim s^{\frac{1}{4}} e^{\mp 2s^{3/2}/3}$  for large values of  $s \approx t(\sim \frac{E}{g^{2/3}})$ . We find that the Eq. (8) becomes independent of energy and it is trivially satisfied for  $g = 0$ . This confirms the Hermitian free particle limit of the soft-box potential wherein any energy is a solution of Eq. (8) and hence discrete energies do not exist.

Thus, we have at least one purely imaginary PT-symmetric potential vanishing asymptotically and possessing real discrete spectrum. This investigation receives significance due to interesting results [15] of Hermitian quantum mechanics like: a real potential well such

that  $\int_{-\infty}^{\infty} V(x)dx < \infty$  has at least one discrete eigen value irrespective of the depth and width of the potential. In the absence of potentials like (7), it would be as though a purely imaginary PT-symmetric potential that vanishes asymptotically does not possess real discrete spectrum.

Earlier we have proposed generation of discrete eigenvalues of a Hermitian potential  $V(x) = -x^4$  [16] by imposing PT-symmetric boundary condition on the wave function. At these discrete energies the reflection probability vanishes and potential becomes reflectionless [16,17]. If we analytically continue  $\kappa$  as  $\kappa = ik$  in the eigenvalue equation (8) we get

$$[kqAi(s) + gAi'(s)][kqBi(t) - gBi'(t)] = [kqBi(s) + gBi'(s)][kqAi(t) - gAi'(t)], k = \sqrt{E}, E > 0. \quad (9)$$

This equation is complex on the real line of  $E$ , so according to *fundamental theorem of algebra* it will have complex roots and some of these roots may or may not be real. For  $E > 0$ , the states are scattering states therein real discrete eigen value/ values if occur would be embedded in positive energy continuum. These states are discrete energy eigenstates generated by imposing PT-symmetric boundary conditions as  $\psi(\pm x) \sim e^{\pm ikx}$ ,  $k = \sqrt{E}$ ,  $E > 0$  instead of the usual  $\psi(\pm x) \sim e^{\pm \kappa x}$ ,  $\kappa = \sqrt{-E}$ ,  $E < 0$ . However, for our soft-box potential (7) we have carried out a very careful search and we do not find real solution(s) of the Eq. (9). Therefore, such reflectionless states do not seem to occur for this potential.

One interesting common feature of  $gx$ -potentials lies in the presence of  $q = (-g^2)^{1/3}$  (cube root of  $-g^2$ ) which has one real ( $q_0$ ) and two complex ( $q_1, q_2$ ) branches. For the Hermitian case, we find that all the three branches of  $q$  give identical discrete spectrum. However for a given eigen value, we have three eigenstates  $\psi_{q_0}(x)$ ,  $\psi_{q_1}(x)$  and  $\psi_{q_2}(x)$ . The first one is real and other two are complex valued functions of  $x$ . Since there can be no degeneracy in one-dimension, we therefore find that the last two eigenstates differ from the first one by multiplicative complex constants only. For the non-Hermitian case of  $igx$ -potentials (1,7), we find that the complex branches  $q_{1,2}$  may not yield the full real discrete spectrum as generated by the main (real) branch  $q_0$ . Only first few eigenvalues of the main branch are repeated. Amusingly, the continuous band of the higher energies turns out to be “eigenvalues” however with null eigenvectors. Every repeated real eigenvalue for the three branches once again yield three corresponding complex eigenfunctions:  $\psi_{q_0}(x)$ ,  $\psi_{q_1}(x)$  and  $\psi_{q_2}(x)$ . Only the first one is also eigenstate of PT, whereas other two are not. We once again find the last two eigenstates differ from the first one by a complex multiplicative factor. This wards off two fake occurrences: degeneracy (in one dimension) and the breaking PT-symmetry (despite

real eigenvalues) when one chooses a complex branch of  $q$ . The complex branches of  $q$  do not create any new energy eigenvalue other than those yielded by the real branch.

Finally, we conclude that though the spectrum of  $V(x) = igx$  is null, however, the spectrum of two of its modifications are interesting. These are semi-analytically solvable models. The PT-symmetric hard-box potential helps studying issues like exceptional points, reality of eigen values, spontaneous PT-symmetry breaking and some interesting features of Hamiltonian-diagonalization. The reduction in the density of *real* eigenvalues as the parameter  $g$  increases is found to be disparate and strong feature of PT-symmetry. We conjecture that PT-symmetry or pseudo-Hermiticity could be a necessary condition for observing exceptional points in the analytically continued complex Hamiltonian. Our soft-box potential is the first solvable model purely imaginary PT-symmetric which vanishes asymptotically and has real discrete spectrum. In the light of this new possibility, it could further be interesting to search for purely imaginary and asymptotically vanishing potentials having discrete spectrum. It may be that the higher-order imaginary potentials of the type  $V_n(x) = iV_2(\text{sech})^n x \tanh x, n > 1$  do possess real discrete spectrum since they will have PT-symmetric turning points of the type  $\pm a + ib$ . Since these are not analytically solvable it would require numerical calculations.

We do not find reflectionless states embedded in positive energy continuum in our soft-box potential.

Very importantly, it turns out that all the three mathematical branches of  $q$  altogether output an unambiguous and non-anomalous physical result (spectrum) in both the cases whether we have Hermitian or PT-symmetric Hamiltonians. Perhaps only  $gx$  or  $igx$  potentials that are amenable to analytic solutions facilitate such a study. This study has not been without interesting surprises as mentioned earlier. Just recently both  $gx$  and  $igx$  potentials have been found very interesting for their complex classical trajectories [18].

We wish that the two modifications of  $igx$ -potential presented here will also be helpful in understanding other issues of PT-symmetric quantum mechanics in future.

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